

GARCH models with spatial structure ^(*)

Modelli GARCH con struttura spaziale

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Riassunto: Questo lavoro presenta una breve rassegna su una nuova classe di modelli GARCH multivariati per la modellizzazione della volatilità di rendimenti azionari, chiamata Spatial Effects in ARCH, SEARCH. La dipendenza spaziale è associata alla classificazione delle imprese in settori industriali e in classi di capitalizzazione. Questa parametrizzazione risulta stimabile anche per ampie dimensioni sezionali, con un numero di parametri che è lineare nella dimensione sezionale, a differenza di modelli alternativi.

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1. Introduction

As it is well recognized in applied econometrics, unrestricted multivariate GARCH models lack parameter parsimony, at least for data-sets with a large cross-section dimension; see e.g. the review paper by Bauwens et al. (2006). In general multivariate GARCH specifications, the number of parameters – i.e. the model dimension – is in fact proportional to the fourth power of the number of assets, see Engle and Kroner (1995). This feature has limited the application of unrestricted GARCH models to systems of limited dimensions, i.e. typically with up to a handful of assets.

We consider a spatial approach in the specification of a GARCH models for stock returns. This model class is called SEARCH, short for Spatial Effects in ARCH; it was proposed and applied in Caporin and Paruolo (2005a, b) – abbreviated CPa and CPb. We provide here a summary of this specification, as well as of the empirical findings.

SEARCH models are designed for possibly large cross-sections of asset returns. Other feasible alternative specifications for large cross-sections include the constant conditional correlation (CC) model of Bollerslev (1990) and the orthogonal GARCH of Alexander (2001), OG. Unlike the CC and OG models, the SEARCH specification has number of parameters that is linear in the cross-section dimension, see CPa. Empirically, see CPb, it was found that the SEARCH specification favorably compares with alternatives.

2. A general class of GARCH models

Consider an $n_x \times 1$ dimensional vector time series $\{x_t\}_{t \in \mathbb{N}}$ and the associated filtration $\mathcal{I}_t := \sigma(x_{t-q}, q \geq 0)$. Let also $x_t := (y_t' : z_t')'$ be partitioned into a $n \times 1$ subvector of

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variables of interest y_t and a $n_z \times 1$ subvector of other information variables z_t . We assume that x_t has finite second moments conditional on $\mathcal{I}_{t-1}$. Indicate by $\mathbb{E}_{t-1}(\cdot) := \mathbb{E}(\cdot | \mathcal{I}_{t-1})$ the conditional expectation operator; let $\mu_t := \mathbb{E}_{t-1}(y_t)$ be the conditional mean of y_t , and let $\varepsilon_t := y_t - \mu_t$ is the $n \times 1$ vector of deviations from the conditional mean.

We consider a parametric model for the conditional mean μ_t , taken for simplicity to be linear, $\mu_t = \mu w_t$, where $w_t := (x'_{t-1} : \dots : x'_{t-p_\mu})'$ is a $n_w \times 1$ vector. We assume that the inter-class dependence of ε_t due to asset proximity can be summarized by

$$\varepsilon_t = S\varepsilon_t + \eta_t, \quad (1)$$

where η_t is a $n \times 1$ vector of random variables with $\mathbb{E}_{t-1}(\eta_t) = 0$ and $\mathbb{V}_{t-1}(\eta_t) = \mathbb{E}_{t-1}(\eta_t \eta_t') =: \Gamma_t$. The matrix $I - S$ plays the role of the matrix of covariance eigenvectors in OG models, see Alexander (2001). When S has spatial structure, (1) defines a spatial autoregression (SAR) process, see Cressie (1993) eq. (6.3.8) and reference therein. In CC models $S = 0$.

We indicate by Σ_t the conditional variance covariance matrix of ε_t , $\Sigma_t := \mathbb{V}_{t-1}(\varepsilon_t) = \mathbb{E}_{t-1}(\varepsilon_t \varepsilon_t')$. A direct consequence of assumption (1) is that

$$\Sigma_t := \mathbb{V}_{t-1}(\varepsilon_t) = (I - S)^{-1} \Gamma_t (I - S')^{-1}.$$

provided $I - S$ is invertible, which we assume in the following. Similarly to the CC class, we assume that the errors η_t in (1) have constant correlations, i.e. $\Gamma_t = D_t R D_t$ where D_t is diagonal and $D_t^2 = \text{diag}(h_t) := \text{diag}(h_{1t}, \dots, h_{nt})$ with (possibly) time varying h_{it} and R is a time-invariant correlation matrix of the standardized innovations $\psi_t := D_t^{-1}(I - S)\varepsilon_t = D_t^{-1}\eta_t$, $\mathbb{V}_{t-1}(\psi_t) = \mathbb{E}_{t-1}(\psi_t \psi_t') = R$.

We consider GARCH(p_E, p_A) dynamics for $h_t := \text{vecd}(D_t^2)$, the vector of diagonal elements of D_t^2 , driven by $e_t := (\eta_{1,t}^2, \dots, \eta_{n,t}^2)'$

$$h_t = \zeta + E(L) h_{t-1} + A(L) e_{t-1}, \quad (2)$$

where by definitions above $h_t = \mathbb{E}_{t-1}(e_t)$, so that $e_t - h_t$ is an innovation with respect to $\mathcal{I}_{t-1}$. This allows to calculate multi-step ahead predictions for the conditional variances using recursions as in standard multivariate GARCH models. Here $E(L) := \sum_{l=1}^{p_E} E_l L^{l-1}$ and $A(L) := \sum_{l=1}^{p_A} A_l L^{l-1}$, E_l, A_l are $n \times n$ matrices, ζ is a n -dimensional vector.

The present class of GARCH models includes the CC and OG as special cases, as well as the SEARCH specification discussed in the following section.

3. Spatial covariance structures

The SEARCH class of models is characterized as the submodel of (1)-(2) with the property that the matrices S, E_l, A_l, R (SEAR) which define the conditional heteroscedasticity (CH) features of the model, have *spatial structure*. Spatial proximity is defined on the basis of the following asset classification scheme.

Let q be the cross-section index in y_{qt} . We assume that there exists a time-invariant classification of the assets by sector and size. In other words we assume that there exist mutually exclusive and exhaustive sets $\mathcal{C}_{ij}$ of indices, such that $q \in \mathcal{C}_{ij}$ indicates that asset number q belongs to sector i and capitalization size j , $i = 1, \dots, k, j = 1, \dots, \ell$.

We define also the aggregated label sets for each sector $\mathcal{C}_i := \cup_{j=1}^{\ell} \mathcal{C}_{ij}$, for each capitalization class $\mathcal{C}_j := \cup_{i=1}^k \mathcal{C}_{ij}$ and for the whole market $\mathcal{C}_\cdot = \cup_{i=1}^k \cup_{j=1}^{\ell} \mathcal{C}_{ij} = \cup_{i=1}^k \mathcal{C}_i = \cup_{j=1}^{\ell} \mathcal{C}_j = \{1, \dots, n\}$. The spatial covariance specification considered in this paper is based on such a classification.

In particular we define spatial weight matrices as follows. We let $W_{ij} := W(\mathcal{C}_{ij}) := (w_{ql})$, where $w_{ql} := (n_{ij} - 1)^{-1} 1(q \in \mathcal{C}_{ij}) (1(l \in \mathcal{C}_{ij}) - \delta_{ql})$ is the normalized spatial matrix that defines as neighbors of the q -th unit the units that belong to the same class $\mathcal{C}_{ij}$; here $n_{ij} := \#(\mathcal{C}_{ij})$ is the cardinality of the set $\mathcal{C}_{ij}$, δ_{ql} is Kronecker's delta and $1(\cdot)$ is the indicator function. This definition is used for $i = \cdot, 1, \dots, k, j = \cdot, 1, \dots, \ell$, i.e. also for the aggregated classes. The matrix S in (1) is specified as

$$S = (s_\cdot W_\cdot) + \left(\sum_{i=1}^k s_i W_i \right) + \left(\sum_{j=1}^{\ell} s_j W_j \right) + \left(\sum_{i=1}^k \sum_{j=1}^{\ell} s_{ij} W_{ij} \right), \quad (3)$$

where each term in parenthesis is interpreted as the effect of a factor with as many levels as elements in the sum. The first one is interpreted as a market factor, the second as a sector factor, the third as a size factor, the last as an interaction factor. The s_{ij} coefficients in (3) are scalar parameters (or possibly diagonal matrices).

Also the GARCH dynamics is restricted to satisfy a spatial specification. The GARCH dynamics is assumed to reflect a possible spatial dependence *within* each class $\mathcal{C}_{ij}$. Without loss of generality, assume that assets are ordered as follows: the first n_{11} assets belong to the class $\mathcal{C}_{11}$, the next n_{12} belong to the class $\mathcal{C}_{12}$, where n_{ij} indicates the number of assets in class $\mathcal{C}_{ij}$.

Consider the partition of e_t into subvectors $e_t := (e'_{11,t}, e'_{12,t}, \dots, e'_{1\ell,t}, e'_{21,t}, \dots, e'_{k\ell,t})'$, where $e_{ij,t}$ is the subvector of e_t corresponding to the class $\mathcal{C}_{ij}$. Partition also η_t, ζ and h_t conformably. We assume that the $n \times n$ matrices E_l, A_l in $E(L) := \sum_{l=1}^p E_l L^{l-1}$ and $A(L) := \sum_{l=1}^q A_l L^{l-1}$ are block diagonal, where blocks are conformable with the partition of e_t . We indicate the blocks of E_l and A_l corresponding to $e_{ij,t-1}$ as $E_{ij,l}$ and $A_{ij,l}$ respectively; similarly we indicate the corresponding blocks of $E(L), A(L)$ as $E_{ij}(L), A_{ij}(L)$.

In order to define a spatial structure on the $E_{ij,l}, A_{ij,l}$ matrices, we assume that assets are ordered within each class $\mathcal{C}_{ij}$ according to a proximity criterion, e.g. by capitalization size. We assume that $E_{ij,l}$ and $A_{ij,l}$ have a spatial structure of the following form:

$$E_{ij,l} := \sum_{q=0}^{m_{E_{ij,l}}} \beta_{ij,lq} W_{ij,q}^* \quad A_{ij,l} := \sum_{q=0}^{m_{A_{ij,l}}} \alpha_{ij,lq} W_{ij,q}^*, \quad (4)$$

where $\beta_{ij,lq}$ and $\alpha_{ij,lq}$ are scalars (or diagonal matrices) and $W_{ij,0}^* := I_{n_{ij}}$. For $q > 0$ the matrices $W_{ij,q}^* := \text{diag}(W_{ij,q}^\circ \iota_{n_{ij}})^{-1} W_{ij,q}^\circ$ are $n_{ij} \times n_{ij}$ normalized spatial weight matrices; here ι_n is a n -vector of ones. Two possible choices for the spatial matrices $W_{ij,q}^\circ$ are $W_{ij,q}^\circ := U_{ij}^{q'}$ or $W_{ij,q}^\circ := U_{ij}^q + U_{ij}^{q'}$ where

$$U_{ij} := \begin{pmatrix} 0 & 0 \\ I_{n_{ij}-1} & 0 \end{pmatrix}. \quad (5)$$

For scalar $\alpha_{ij,lq}$ and $\beta_{ij,lq}$ parameters, this choice of $W_{ij,q}^\circ$ implies that $E_{ij,l}$ and $A_{ij,l}$ are Toeplitz matrices.

These specifications have the following interpretation: the spatial structure within each block relates each stock within $e_{ij,t}$ with the preceding one in the list; this is the case for specification $W_{ij,q}^\circ := U_{ij}^{q'}$, which implies an upper triangular system. Alternatively, for specification $W_{ij,q}^\circ := U_{ij}^q + U_{ij}^{q'}$, each stock within $e_{ij,t}$ is related to the one preceding and the one following it in the list, which implies a symmetric Toeplitz matrix for the scalar spatial coefficients.

The conditions for positive definiteness of the conditional variance matrix require the EA dynamics to deliver always positive definite conditional variances h_t . A sufficient condition for this is that the ζ , α and β parameters are positive.

Also the correlation matrix R is assumed to be block diagonal with diagonal blocks R_{ij} , where the subscripts ij refer to the class $\mathcal{C}_{ij}$. R_{ij} describes intra-class correlations within class $\mathcal{C}_{ij}$. We consider the following spatial specification:

$$R_{ij} = I_{n_{ij}} + \sum_{q=1}^{m_{R_{ij}}} \rho_{ij,q} W_{ij,q}^* \quad (6)$$

where $W_{ij,q}^\circ := U_{ij}^q + U_{ij}^{q'}$, $W_{ij,q}^* := \text{diag}(W_{ij,q}^\circ \iota_{n_{ij}})^{-1} W_{ij,q}^\circ$ and U_{ij} is defined in (5) above. A special case is given by the spatial autoregression of order one, where $\rho_{ij,q} = \rho_{ij}^q$ for some scalar ρ_{ij} . Another special case of interest is given by the restriction $\rho_{ij,q} = \rho_{ij}$ for some scalar ρ_{ij} and all q . This corresponds to the spatial classification of all the assets within class $\mathcal{C}_{ij}$ as first-order neighbors. The latter case is found to be empirically supported in CPb on a cross-section of 150 assets from the New York stock exchange.

A number of possible extensions and variations can be considered on the basic scheme proposed here, see CPa, who also discuss the relation of the SEARCH model with other GARCH specifications.

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